Sequential detection of the order of integration for pth-order autoregressive model

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Introduction

For a pth-order autoregressive (AR(p)) model, we propose a sequential detection procedure of the order d of integration (I(d)). We introduce three stopping times and two t-statistics. Simulation studies are conducted for sequential detection of order $d = 0, 1, 2$ of integration for AR(3) models to verify our theory.

Model, stopping times and t-statistics

Suppose we observe the following AR(p) process $\{x_n\}$ sequentially

$$(1 - \alpha_1 L)(1 - \alpha_2 L) \cdots (1 - \alpha_p L)x_n = \epsilon_n, \quad n = 1, \ldots, \tag{1}$$

where L is the lag operator and $\epsilon_n \sim i.i.d.\ (0, \sigma^2)$. Without loss of generality, we assume $|\alpha_i| \leq |\alpha_2| \leq |\alpha_1| \leq 1$ $(i = 3, \ldots, p)$. We define three stopping times

$$\tau_k = \inf\left\{N : \frac{\hat{\Psi}_k^{\ 2}(1)}{cs^2} \sum_{t=p+1}^{N} x_{t-1}^2 \geq 1\right\}\ (k = 1, 2), \tau_\Delta = \inf\left\{N : \frac{\hat{\Psi}_2^2(1)}{cs^2} \sum_{t=p+1}^{N} \Delta x_{t-1}^2 \geq 1\right\}, \tag{2}$$

where s^2 is a consistent estimator of σ^2 and $\hat{\Psi}_1(1), \hat{\Psi}_2(1)$ are consistent estimators of $\Psi_1(1) = \Pi_{i=2,\ldots p}(1 - \alpha_i)$ and $\Psi_2(1) = \Pi_{i=3,\ldots p}(1 - \alpha_i)$ respectively. Using stochastic calculus in continuous time, as $c \to \infty$,

$$\frac{\tau_1}{\sqrt{c}} \Rightarrow U_1 = \inf\left\{t : \int_0^t W_u^2 du = 1\right\} \quad \text{if } \{x_n\} \textit{ is an } I(1) \text{ process} \tag{3}$$

$$\frac{\tau_2}{c^{1/4}} \Rightarrow V_1 = \inf\left\{t : \int_0^t F_u^2 du = 1\right\} \quad \text{if } \{x_n\} \textit{ is an } I(2) \text{ process} \tag{4}$$

where W_t is a standard Brownian motion and $F_t = \int_0^t W_u du$. In our companion paper, we have known the distribution of U_1 in (3) and V_1 in (4). The two test statistics is defined as

$$J = \frac{\hat{\Psi}_1(1)}{\sqrt{c}s_N^2} \sum_{n=4}^{\tau_1} x_{n-1}\left\{\hat{\Psi}_1(L)\Delta x_n\right\} \quad, J_\Delta = \frac{\hat{\Psi}_2(1)}{\sqrt{c}s_N^2} \sum_{n=4}^{\tau_\Delta} \Delta x_{n-1}\left\{\hat{\Psi}_2(L)\Delta^2 x_n\right\}. \tag{5}$$

Detection Method

Let u_q, v_q and z_q be the $q \times 100$ percentile of U_1, V_1 and the standard normal distribution. To compute the value of v_q, we use the results in Tanaka(2017). We start from $I(2)$. Let $\alpha = 0.01$ for example.

1. If $\tau_2/c^{1/4} < v_{1-\alpha}$, reject $I(0)$. Then we use J_Δ for testing $I(1)$ and $I(2)$
   (a) If $J_\Delta < z_\alpha$, we choose $I(1)$. (b) If $J_\Delta \geq z_\alpha$, we choose $I(2)$.
2. If $\tau_2/c^{1/4} \geq v_{1-\alpha}$, we reject $I(2)$. Then we use $\tau_1/\sqrt{c}$ for testing $I(0)$ and $I(1)$.
   (a) If $\tau_1/\sqrt{c} > u_{1-\alpha}$, we choose $I(0)$.
   (b) If $\tau_1/\sqrt{c} \leq u_{1-\alpha}$, then we use J for testing $I(0)$ and $I(1)$.
   If $J < z_\alpha$, select $I(0)$. If $J \geq z_\alpha$, select $I(1)$.

Reference

1. Tanaka. K.(2017). Time series analysis: Nonstationary and Noninvertible Distribution Theory, 2nd edition. Wiley