Bayesian predictive distributions for Poisson and negative binomial models when the parameter spaces are restricted

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Suppose that X and Y are independent Poisson random variables with means $r\lambda$ and $s\lambda$, respectively, and that $\lambda > 0$ is the unknown parameter while r and s are known positive constants. Suppose further that the parameter space of λ is restricted to the interval $(0, \overline{\lambda}]$ for some known constant $\overline{\lambda} > 0$. We first compare the Bayes estimator of λ based on the observation of X under the prior $\pi_\beta(\lambda) = \lambda^{\beta-1}$, $\beta > 0$, with that under the truncated prior $\pi_{\beta,\overline{\lambda}}(\lambda) = \pi_\beta(\lambda) \cdot 1_{(0,\overline{\lambda}]}(\lambda)$ and give a sufficient condition for the latter to improve on the former under the loss function

$$L(d, \lambda) = d - \lambda - \lambda \log(d/\lambda). \qquad (*)$$

Next we translate this result into the framework of predictive probability estimation, where the problem is to construct a predictive probability function $\widehat{p}(y; X)$ for the probability function of the future observation Y, $p_s(y|\lambda) = (s^y/y!)\lambda^y e^{-s\lambda}$, on the basis of the current observation X. Utilized in this approach is an identity derived in the literature that relates Bayesian predictive probability estimation to Bayesian point estimation in the Poisson case.

For the negative binomial case, we consider a similar problem and use a similar strategy by deriving a result that relates estimating a negative binomial probability function to estimating powers of a negative binomial failure probability. The result is as follows. Let $f_r(x|p) = [\Gamma(r+x)/\{\Gamma(r)\Gamma(x+1)\}]p^r(1-p)^x$, $x = 0, 1, 2, \ldots$, be the probability mass function of the negative binomial distribution with size r and probability p. Suppose that X and Y are independently distributed with probability functions $f_r(x|p)$ and $f_s(y|p)$, respectively, and that $p \in (0, 1)$ is unknown while $r, s > 0$ are known. Let $\pi(p)$ be a prior density for p and let $\widehat{f}^{(\pi)}(y; X)$ be the posterior mean of $f_s(y|p)$ with respect to the prior $\pi(p)$ and the observation of X. Then the expectation of the Kullback-Leibler divergence of $\widehat{f}^{(\pi)}(y; X)$ from $f_s(y|p)$, denoted $R(p, \widehat{f}^{(\pi)})$, can be expressed as

$$R(p, \widehat{f}^{(\pi)}) = \int_r^{r+s} \sum_{n=1}^{\infty} \frac{1}{n} E_p^{Z_t}[L(\widehat{\theta}_{n,t}^{(\pi)}(Z_t), \theta_n)]dt,$$

where $L(\cdot, \cdot)$ is the loss function given in $(*)$, $\theta_n = (1-p)^n$ for $n = 1, 2, \ldots$, and, for $t \in [r, r+s]$, Z_t is a negative binomial random variable with probability mass function $f_t(z|p)$ and $\widehat{\theta}_{n,t}^{(\pi)}(Z_t) = E_\pi^{p|Z_t}[(1-p)^n|Z_t]$, the posterior mean of θ_n given Z_t under the prior $\pi(p)$.

References

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