On Nonparametric Estimation of Dilatation

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Our problem is to evaluate a harmonized relationship between two d-dimensional variables X and Y using the data $(X_1, Y_1), \ldots, (X_N, Y_N)$ of size N. Here a harmonized relationship is defined as the existence of a certain conformal mapping between X and Y: a transformation is harmonious if it is a conformal mapping. Here we introduce some mathematical symbols and the definition of *quasiconformal*. For $x = (x_1 \cdots x_d)^T \in \mathbb{R}^d$, the norm of x is defined as $|x| = \sqrt{x^T x}$, where ^{T} designates the transpose. The operator norm of a $d \times d$ matrix A is defined as

$$||A|| = \sup\{|Ay| : y \in \mathbb{R}^d, |y| = 1\}.$$

Let g be a transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$: $g(x) = [g_1(x) \ \cdots \ g_d(x)]^T$, $x \in \mathbb{R}^d$, where each g_i is a smooth function from $\mathbb{R}^d$ to $\mathbb{R}$. The Jacobi matrix of g at x is defined as

$$Dg(x) = \left(\frac{\partial g_i(x)}{\partial x_j}\right)_{1 \le i,j \le d}$$

and the modulus of its determinant is denoted as $J_g(x) = |\det(Dg(x))|$. Let Δ be a domain in $\mathbb{R}^d$ and let f be a homeomorphism from Δ to $\mathbb{R}^d$ with $J_f(x) \neq 0$ on Δ.

Definition *The homeomophism f is said to be K-quasiconformal if and only if there exists a $K > 0$ such that $||Df(x)||^d \le K \cdot J_f(x)$ for all x in Δ.*

Conformal mapping is known to be characterized by 1-quasiconformal. Now by introducing

$$\kappa(x|f) = \frac{||Df(x)||^d}{J_f(x)},$$

we have another expression for the definition of K-quasiconformal: f is K-quasiconformal if and only if there exists a $K > 0$ such that $\kappa(x|f) \le K$ for all x in Δ. The general definition of dilatation $d(f)$ of the transformation f is

$$d(f) = \max_{x \in \Delta} \kappa(x|f).$$

Notice that $d(f) \ge 1$ always holds, and $d(f) = 1$ is equivalent to that f is conformal, hence representing a harmonized relationship. Therefore the dilatation $d(f)$ can be utilized as a measure to evaluate the discrepancy between f and harmonious transformations.

In this talk we consider nonparametric kernel estimation of the dilatation using the data $(X_1, Y_1), \ldots, (X_N, Y_N)$. Asymptotic results as well as numerical results will be presented.