

# Bradley-Terry model and its related topics

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The Bradley-Terry model was first proposed to study paired comparisons. Let  $a$  be the number of items in a preference test. After all paired comparisons  $(i, j)$ ,  $i, j = 1, 2, \dots, a$ ,  $i < j$ , are done, we may estimate the ranking of items by using the above preference test scores. Bradley and Terry proposed  $p_{ij} = \delta_i / (\delta_i + \delta_j)$ , where  $p_{ij}$  represents the probability that item  $i$  is preferred rather than item  $j$  in the preference testing in the matched pair  $(i, j)$ .

Later, the Bradley-Terry model was used to evaluate the ability of players based on athletic competitions. Let  $F(x)$  be any continuous distribution function and  $X_i$  be the ability of player  $i$  in the competition. Assume that the distribution function of  $X_i$  is  $P(X_i \leq x) = F^{\delta_i}(x)$ ,  $i = 1, 2, \dots$ , where  $\delta_i > 0$ . Assume also  $X_i$  and  $X_j$  are independent. Then  $P(X_i > X_j) = \iint_{x_i > x_j} dF^{\delta_i}(x_i) dF^{\delta_j}(x_j) = \delta_i / (\delta_i + \delta_j)$ .

Another way of approaching the Bradley-Terry model is described in Takeuchi (1980). If  $p_{ij}$  which is the probability that Player  $i$  wins in a match between Player  $i$  and Player  $j$ , satisfies  $p_{ij}p_{jk}p_{ki} = p_{ik}p_{kj}p_{ji}$ ,  $p_{ij}$  is expressed as  $p_{ij} = \delta_i / (\delta_i + \delta_j)$ .

In this presentation, the above results will be extended to the case of three or more player competitions.

When singles and doubles matches are played in some sports competitions such as tennis, it is reasonable to assume that the pair of Players  $(i, j)$  has the ability of  $\delta_i + \delta_j$ . Then, the probability which Pair  $(i, j)$  defeats Pair  $(k, l)$  in a doubles game is expressed as  $p_{(ij),(kl)} = (\delta_i + \delta_j) / (\delta_i + \delta_j + \delta_k + \delta_l)$ . This presentation also discusses how to estimate  $\delta_i$ , *i.e.*, the ability of Player  $i$ , by using data from both singles and doubles. Some related problems are also discussed.

**References** [1] Bradley, R. and Terry, M. (1952). Rank analysis of incomplete block designs I. The method of paired comparisons. *Biometrika*, **39**, 324-345. [2] Bradley, R. A. (1976). Science, statistics, and paired comparisons. *Biometrics*, **32**, 213-240. [3] Hirotsu, C. (1982). *Analysis of Discrete Data*, (in Japanese) Kyôiku-shuppan. [4] Takeuchi, K. (1980). *Statistical Science for Phenomenon and Behavioral Science*, (in Japanese) Shinyo-sha. [5] Takeuchi, K. and Fujino, Y. (1988). *Mathematical Statistics for Sports*, (in Japanese) Kyôritsu-shuppan.