# A note on Stein's method for F-distribution

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Stein's method is a unified method of determining accuracy of the approximation of one probability distribution by another. Stein's method for a target distribution $\mu$ can be described as follows . (1) Find an operator $\mathcal{A}$ such that $X \sim \mu$ if and only if for all smooth functions $f$, $\mathbb{E}\mathcal{A}f(X) = 0$ holds. (2) Find a solution $f_h$ of the Stein equation

$$h(x) - \int hd\mu = \mathcal{A}f(x). \tag{1}$$

(3) Then for any variable $W$, it follows that $\mathbb{E}h(W) - \int hd\mu = \mathbb{E}\mathcal{A}f(W)$.

In this note Stein's method for $F$-distribution is investigated. A characterization of $F$-distribution can be derived by the differential equation approach described in Döbler (2012). Let $p(x)$ be the density function of $F(d_1, d_2)$. The mean of $F(d_1, d_2)$ is $d_2/(d_2 - 2)$ when $d_2 > 2$. It is easy to see that the density $p$ satisfy the differential equation

$$(2d_1x^2 + 2d_2x)p'(x) + \{d_1(d_2 + 2)x + 2d_2 - d_1d_2\}\, p(x) = 0 \tag{2}$$

Let $\alpha(x) = (2d_1x^2 + 2d_2x)$ and $\beta(x) = \{d_1(d_2 + 2)x + 2d_2 - d_1d_2\}$.

It is shown that a random variable $X$ is distributed according to $F(d_1, d_2)$ if and only if

$$\mathbb{E}\left[-2d_1X^2f'(X) - 2d_2Xf'(X) + d_1(d_2 - 2)Xf(X) - d_1d_2f(X)\right] = 0 \tag{3}$$

for any $f$ for which above expected value is finite.

Consider Stein equation

$$\eta(x)f'(x) + \gamma(x)f(x) = \tilde{h}(x) \tag{4}$$

where $\eta(x) = \frac{2d_2}{d_2-2}x\left(1 + \frac{d_1}{d_2}x\right)$ and $\gamma(x) = d_1\left(\frac{d_2}{d_2-2} - x\right)$ and $\tilde{h}(x) = h(x) - Fh$. Here $Fh$ denotes $\mathbb{E}(Z)$ where $Z$ is distributed according to $F(d_1, d_2)$. The solution $f_h$ of (5) and smoothness estimates of $f_h$ are obtained by the arguments similar to those in section 2.5 of Döbler(2012).

**Lemma 1.** *Let $f_h$ be the solution of Stein equation (5). Then, if $h : \mathbb{R}^+ \to \mathbb{R}$ is bounded,*

$$\|\eta(x)f_h'(x)\| \le 2\|h - F_{d_1,d_2}h\| \le 4\|h\|$$

**Theorem 1.** *If $h$ is Lipschitz,then*

$$\|f_h(x)\|_\infty \le \frac{1}{d_1}\|h\|_\infty \tag{5}$$

*and there exists a positive constant $C(d_1, d_2)$ such that*

$$\|f_h'(x)\|_\infty \le C(d_1, d_2)\|h'\|_\infty. \tag{6}$$

# References

[1] Döbler, C. (2015). Stein's method of exchangeable pairs for the Beta distribution and generalizations. Electron. J. Probab. Volume 20, paper no. 109, 34 pp.

[2] Gaunt, Mijoule and Swan (2016). arXiv:1604.06819. Stein operators for product distributions.